Anomalous scalings for fluctuations of inertial particles concentration and large-scale dynamics

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Small-scale fluctuations and mean-field dynamics of the number density of inertial particles in turbulent fluid flow are studied. Anomalous scaling for the second-order correlation function of the number density of inertial particles is found. The mechanism for the anomalous scaling is associated with the inertia of particles that results in a divergent velocity field of particles. The anomalous scaling appears already in the second moment when the degree of compressibility $\sigma > 1/27$ (where σ is the ratio of the energies in the compressible and the incompressible components of the particles velocity). The δ-correlated in time random process is used to describe a turbulent velocity field. However, the results remain valid also for the velocity field with a finite correlation time, if all moments of the number density of the particles vary slowly in comparison with the correlation time of the turbulent velocity field. The mechanism of formation of large-scale inhomogeneous structures in spatial distribution of inertial particles advected by a low-Mach-number compressible turbulent fluid flow with a nonzero mean temperature gradient is discussed as well. The effect of inertia causes an additional nondiffusive turbulent flux of particles that is proportional to the mean temperature gradient. Inertial particles are concentrated in the vicinity of the minimum (or maximum) of the mean temperature of the surrounding fluid depending on the ratio of the material particle density to that of the surrounding fluid. The equation for the turbulent flux of particles advected by a low-Mach-number compressible turbulent fluid flow is derived. The large-scale dynamics of inertial particles is studied by considering the stability of the equilibrium solution of the derived equation for the mean number density of the particles. A modified Rayleigh-Ritz variational method is used for the analysis of the large-scale instability. [S1063-651X(98)04209-3]

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I. INTRODUCTION

Turbulent transport of passive scalar (e.g., number density of particles) advected by an incompressible fluid flow in the case $\mathbf{U}=\mathbf{v}$ has been studied in a large number of publications. Here $\mathbf{U}$ is the particle velocity and $\mathbf{v}$ is the velocity of the surrounding fluid.

Interesting features in turbulent transport of passive scalar appear when $\mathbf{U}\neq\mathbf{v}$. In this case certain phenomena, e.g., turbulent thermal diffusion [1], turbulent barodiffusion [2], and self-excitation, i.e., exponential growth of fluctuations of the number density of inertial particles [3] occur. These effects are caused by inertia of particles that results in a divergent velocity field of particles. The self-excitation of fluctuations of the number density of particles results in the intermittency in spatial distribution of inertial particles [3]. Notably, small-scale inhomogeneities in the spatial distribution of inertial particles were observed in the laboratory [4] and in the atmospheric turbulent flows [5,6].

There are two types of intermittency of passive scalar distribution: the intermittency in the systems with and without external pumping. Intermittency in the systems without external pumping was predicted by Zeldovich, Molchanov, Ruzmaikin, and Sokoloff in 1985 (see, e.g., [7–9]). In these systems under certain conditions there is a self-excitation of fluctuations of passive scalar or vector (magnetic) fields. The growth rate γ_s of the s-order correlation function of passive scalar (see [3]) and vector (see [9,10]) fields is given by $\gamma_s \sim s^2\gamma_2$ for $s\gg 1$ [where γ_2 is the growth rate of the second-order correlation function of passive scalar or vector (magnetic) fields]. This implies that when $\gamma_2>0$ (i.e., fluctuations of passive fields are excited), higher moments grow faster than lower moments, i.e., $\gamma_s>\gamma_{s-1}$ and $\gamma_s>s\gamma_2/2$. This results in intermittency, i.e., the appearance of sharp peaks in which the main part of the field intensity is concentrated. Fluctuations of vector (magnetic) can be excited even by incompressible turbulent three-dimensional (3D) flows of conducting fluid (see, e.g., [9,10]). On the other hand, passive scalar fluctuations (e.g., fluctuations of the particles number density) can be excited when $\mathbf{U}\neq\mathbf{v}$ (e.g., for inertial particles) or when $\mathbf{U}=\mathbf{v}$ but div $\mathbf{v}\neq 0$ (e.g., for a low-Mach-number compressible turbulent fluid flow) [3].

Intermittency in the systems with external pumping was predicted by Kraichnan in 1991 (see, e.g., [11]). Here the fluctuations of passive scalar are sustained by an external source. In these systems a problem of anomalous scaling arises. The anomalous scaling means the deviation of the scaling exponents of the correlation function of a passive field from their values obtained by the dimensional analysis. For incompressible turbulent flow and when $\mathbf{U}=\mathbf{v}$ the anomalous scalings for scalar field can occur only beginning with a forth-order correlation function (see e.g., [12–14]) while for the vector (magnetic) field the anomalous scalings appear already in the second moment [10,15].

In the present paper we show that the anomalous scalings appear already in the second-order correlation function of the number density of inertial particles when the degree of compressibility $\sigma>1/27$ (where σ is the ratio of the energies in the compressible and the incompressible components of the particles velocity). In this case there is no self-excitation of fluctuations of the number density of inertial particles and these fluctuations are sustained by an external source. We use here for simplicity the δ-correlated in time random process to describe a turbulent velocity field. However, the results remain valid also for the velocity field with a finite correlation time, if all moments of the number density of the particles vary slowly in comparison with the correlation time of the turbulent velocity field (see, e.g., [16]).

Which type of intermittency can occur in a system? It depends on the Reynolds number in the case of inertial particles and on the magnetic Reynolds number in the case of magnetic fluctuations. When the Reynolds number is larger than a certain critical value the first type of intermittency (without external pumping) occurs [3]. On the other hand, when the Reynolds number is smaller than that the certain critical value the second type of intermittency (with external pumping) appears.

When $\mathbf{U}\neq\mathbf{v}$ the dynamics of the mean number density of inertial particles are strongly changed. In the present paper we study large-scale dynamics of inertial particles in a low-Mach-number compressible turbulent fluid flow. We discuss here a mechanism of formation of large-scale inhomogeneous structures in spatial distribution of inertial particles advected by a turbulent fluid flow with a nonzero mean temperature gradient. This effect is caused by both a divergent particles velocity field of and the correlation between temperature and velocity fluctuations of the surrounding fluid [1,2]. This phenomenon results in a relatively strong nondiffusive mean flux of inertial particles in regions with mean temperature gradients. Under certain conditions the initial spatial distribution of small inertial particles evolves into a highly inhomogeneous large-scale pattern where domains with increased particles concentration border on domains depleted of particles.

In this paper we have derived an equation for the turbulent flux of particles advected by a low-Mach-number compressible turbulent fluid flow. In our previous study [1] only the case with very small compressibility of fluid flow (i.e., $|\mathrm{div}\,\mathbf{v}|\ll|\mathrm{div}\,\mathbf{U}|$) was considered. The large-scale dynamics are studied by considering the stability of the equilibrium solution of the derived evolution equation for the mean number density of the particles for large Péclet numbers. The resulting equation is reduced to an eigenvalue problem for a Schrödinger equation with a variable mass, and a modified Rayleigh-Ritz variational method is used to estimate the lowest eigenvalue corresponding to the growth rate of the instability.

II. SMALL-SCALE FLUCTUATIONS

Number density $n_p(t,\mathbf{r})$ of small particles in a turbulent flow is determined by the equation:

$$\frac{\partial n_p}{\partial t}+\vec{\nabla}\cdot(n_p\mathbf{U})=D\Delta n_p, \tag{1}$$

where $\mathbf{U}$ is a random velocity field of the particles that they acquire in a turbulent fluid velocity field, and D is the coefficient of molecular diffusion. We consider the case of large Reynolds and Péclet numbers.

To study the fluctuations of inertial particle concentration we derive equation for the second-order correlation function of particle concentration. For this purpose we use a method of path integrals (Feynman-Kac formula) that has been previously employed in magnetohydrodynamics [8–10,17] and in the problems of passive scalar transport in incompressible [8,9] and compressible [1–3,18] turbulent flows. The use of this technique allows us to derive an equation for the second-order correlation function $\Phi=\langle\Theta(\mathbf{x})\Theta(\mathbf{y})\rangle$:

$$\frac{\partial\Phi}{\partial t}=-2[D_{mn}(0)-D_{mn}(\mathbf{r})]\frac{\partial^2\Phi}{\partial x_m\partial y_n}+2\langle\tau b(\mathbf{x})b(\mathbf{y})\rangle\Phi-4\langle\tau u_m(\mathbf{x})b(\mathbf{y})\rangle\frac{\partial\Phi}{\partial x_m}+I \tag{2}$$

(see Appendix A), where $\Theta=n_p-N$, $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $I=2\langle\tau b(\mathbf{x})b(\mathbf{y})\rangle N^2$, and $D_{pm}=D\delta_{pm}+\langle\tau u_p u_m\rangle$, and $N=\langle n_p\rangle$ is the mean number density of particles, $\mathbf{U}=\mathbf{V}_p+\mathbf{u}$, and $\mathbf{V}_p=\langle\mathbf{U}\rangle$ is the mean particles velocity, $b=\vec{\nabla}\cdot\mathbf{u}$, and τ is the momentum relaxation time of a random velocity field $\mathbf{u}$, which depends on the scale of turbulent motion. We use here for simplicity the δ-correlated in time random process to describe a turbulent velocity field. However, the results remain valid also for the velocity field with a finite correlation time, if all moments of the number density of the particles vary slowly in comparison with the correlation time of the turbulent velocity field (see, e.g., [16]).

Using the δ-correlated in time random process allows us to provide the analytical calculations and to obtain closed results for the growth rate of the second-order correlation functions of the particles number density, the threshold of the generation of the passive scalar fluctuations and their anomalous scaling. The use of the δ-correlated in time random process to describe a turbulent velocity field is an approximation. However, we study here two specific problems: (i) Conditions for the self-excitations of the fluctuations of the particles number density (i.e., the threshold of the generation and growth rate of the the passive scalar fluctuations in the vicinity of the threshold); (ii) the anomalous scaling behavior that is determined by the ''zero mode'' of the equations for correlation functions of the particles number density (''zero mode'' is a mode with zero growth rate). In the vicinity of the threshold the characteristic time of variations of the high-order correlation functions of the particles number density is much larger than the momentum relaxation time $\tau(\mathbf{r})$. The latter allows us to use the δ-correlated in time random process to describe a turbulent velocity field.

Equation (2) for $b=0$ was first derived by Kraichnan (see [19]). In this particular case, $b=0$, this equation describes a relaxation of the second moment of particles number density. On the other hand, when $b\neq0$, i.e., when the velocity of particle is divergent, Eq. (2) implies both, an effect of self-excitation (exponential growth) of fluctuations of particles number density caused by the second term in Eq. (2) (see [3]) and anomalous scalings for the fluctuations (see Sec. III). Another interesting feature of Eq. (2) is the emergence of the ''internal'' source term $I=2\langle\tau b(\mathbf{x})b(\mathbf{y})\rangle N^2$. The latter means that external pumping is not required in order to sustain the fluctuations even when there is no self-excitation of the fluctuations of particles number density [3].

We consider a homogeneous and isotropic turbulent velocity field of fluid. In this case the particle velocity field is also homogeneous and isotropic, and it is compressible, i.e.,

$\vec{\nabla}\cdot\mathbf{U}\neq 0$. Indeed, the velocity of particles $\mathbf{U}$ depends on the velocity of the surrounding fluid $\mathbf{v}$, and it can be determined from the equation of motion for a particle:

$$d\mathbf{U}/dt=(\mathbf{v}-\mathbf{U})/\tau_p+\mathbf{F}/m_p, \tag{3}$$

where τ_p is the characteristic time of coupling between the particle and surrounding fluid (Stokes time), $\mathbf{F}$ is the external force, and m_p is the mass of particles.

A solution of the equation of motion for particles can be written in the form $\mathbf{U}=\mathbf{v}+\tau_p\mathbf{f}(\mathbf{v},\tau_p)$. The second term in this solution describes the difference between the local fluid velocity and particle velocity arising due to the small but finite inertia of the particle. We calculate the divergence of the equation of motion for particles, and after simple manipulation we obtain

$$\vec{\nabla}\cdot\mathbf{U}=-\tau_p\vec{\nabla}\cdot[(\mathbf{v}\cdot\vec{\nabla})\mathbf{v}]-\tau_p^2\vec{\nabla}\cdot\left[\frac{\partial\mathbf{f}}{\partial v_k}\frac{dv_k}{dt}\right]. \tag{4}$$

When τ_p is very small Eq. (4) coincides with the results obtained in [20]. The Navier-Stokes equation for the fluid yields $\vec{\nabla}\cdot[(\mathbf{v}\cdot\vec{\nabla})\mathbf{v}]=-\Delta P_f/\rho$, where P_f is the pressure of a fluid. From the latter equation and Eq. (4) it is seen that $\vec{\nabla}\cdot\mathbf{U}\neq 0$.

The equation of motion (3) for a particle is valid when the density of surrounding fluid ρ is much less than the material density ρ_p of particles ($\rho\ll\rho_p$) [21,22]. However, the results of the study can be easily generalized to include the case $\rho\gg\rho_p$ using the equation of motion of particles in fluid flow presented in [21,22]. This equation of motion takes into account contributions due to the pressure gradient in the fluid surrounding the particle (caused by acceleration of the fluid) and the virtual (''added'') mass of the particles relative to the ambient fluid. Solution of this equation for small τ_p coincides with the solution of Eq. (3) except for the transformation $\tau_p\to\beta_*\tau_p$, where

$$\beta_*=\left(1+\frac{\rho}{\rho_p}\right)\left(1-\frac{3\rho}{2\rho_p+\rho}\right).$$

The correlation function of a compressible homogeneous and isotropic random velocity field was derived in [23]. The second moment for the particle velocity can be chosen in the same form (see below):

$$\langle\tau u_m(\mathbf{x})u_n(\mathbf{x}+\mathbf{r})\rangle=D_T\bigg\{[F(r)+F_c(r)]\delta_{mn}+\frac{rF'}{d-1}\left(\delta_{mn}-\frac{r_m r_n}{r^2}\right)+rF_c'\frac{r_m r_n}{r^2}\bigg\} \tag{5}$$

(for details see [23]), where $F'=dF/dr$, $F(0)=1-F_c(0)$, and $D_T=u_0l_0/d$, and l_0 is the maximum scale of turbulent motions, u_0 is the characteristic velocity in this scale, and d is the dimensionality of space. The function $F_c(r)$ describes the potential component whereas $F(r)$ corresponds to the vortical part of the turbulent velocity of particles.

We seek a solution to the equation for Φ without the external source I in the form

$$\Phi(t,r)=\Psi(r)r^{(1-d)/2}\exp\left[-\int_0^r\chi(x)dx\right]\exp(\gamma t). \tag{6}$$

Substituting Eq. (6) into Eq. (2) yields an equation for unknown function $\Psi(r)$:

$$\frac{1}{m(r)}\Psi''-[\gamma+U_0(r)]\Psi=0, \tag{7}$$

where

$$U_0(r)=\frac{1}{m(r)}\left[\frac{d-1}{r}\chi+\chi^2+\chi'+\frac{(d-1)(d-3)}{4r^2}\right]-\kappa(r), \tag{8}$$

$$\frac{1}{m(r)}=\frac{2}{\mathrm{Pe}}+\frac{2}{d}[1-F-(rF_c)'], \tag{9}$$

$$\chi(r)=m(r)[(3d+1)F_c'-F'+2rF_c'']/d, \tag{10}$$

$$\kappa(r)=-2[(d^2-1)F_c'/r+(2d+1)F_c''+rF_c''']/d, \tag{11}$$

and distance r is measured in units of l_0, time t is measured in units of $\tau_0=l_0/u_0$, and $\mathrm{Pe}=l_0u_0/D\gg 1$ is the Péclet number.

Now we discuss the above model of a random velocity field of inertial particles. Consider a case when $\tau_\nu\ll\tau_p\ll\tau_0$, and the particle radius $a_*\ll l_\nu$, where τ_ν is the correlation time in the viscous dissipation scale l_ν of a fluid flow. The viscous scale is $l_\nu\sim\mathrm{Re}^{-1/(3-p)}$, where $\mathrm{Re}=l_0u_0/\nu_0\gg 1$ is the Reynolds number, ν_0 is the kinematic viscosity of the fluid, and p is the exponent in the spectrum of the turbulent kinetic energy of fluid. Consider the case when the material density ρ_p of particles is much larger than the density ρ of fluid. Introduce a scale r_a in which $\tau_p=\tau(r=r_a)$, where $l_\nu\ll r_a\ll 1$, and $\tau(r)$ is the correlation time of the turbulent fluid velocity field in the scale r. In the range $r_a\ll r<1$ the effect of inertia of particles is very small and particles velocity is close to the fluid velocity. In this case $F=1-r^{q-1}+O(\tau_p^2/\tau_0^2)$ and $F_c=O(\tau_p^2/\tau_0^2)$, where $q=2p-1<3$. Note that the exponent p in the spectrum of kinetic turbulent energy is different from that of the function $\langle\tau u_m u_n\rangle$ due to the scale dependence of the momentum relaxation time τ of turbulent velocity of fluid [10]. Thus in the scales $r_a\ll r<1$ the effects of compressibility of the particles velocity field is negligible.

On the other hand, in scales $l_\nu\ll r<r_a$ the effect of inertia is important so that $\vec{\nabla}\cdot\mathbf{U}\neq 0$. In these scales incompressible $F(r)$ and compressible $F_c(r)$ components of the turbulent velocity field of particles can be chosen as $F(r)=(1-\varepsilon)(1-r^{q-1})$, and $F_c(r)=\varepsilon(1-r^{q-1})$. We take into account that in the equation of motion for particles $d\mathbf{U}/dt=(\mathbf{v}-\mathbf{U})/\tau_p$ the last term $|\mathbf{U}/\tau_p|\ll|d\mathbf{U}/dt|$ in the scales $l_\nu\ll r<r_a$. In this case the equation of motion for particles coincides with the Navier-Stokes equation for fluid in the inertial range (where the viscous term is dropped out) except for the term $\propto\vec{\nabla}P$. In the latter equation for particle motion

the term $\mathbf{v}/\tau_p$ can be interpreted as a stirring force. Thus in this case it is plausible to suggest that the exponent in the spectrum of the second moment of particle velocity coincides with that of the turbulent fluid velocity. However, $|\vec{\nabla}\cdot\mathbf{U}|\propto|\vec{\nabla}\cdot[(\mathbf{U}\cdot\vec{\nabla})\mathbf{U}]|\neq 0$.

In scales $0\leqslant r<l_\nu$ incompressible $F(r)$ and compressible $F_c(r)$ components of the turbulent velocity field of particles are given by $F(r)=(1-\varepsilon)(1-\alpha r^2)$, and $F_c(r)=\varepsilon(1-\alpha r^2)$, where $\alpha=\mathrm{Re}^{(3-q)/(3-p)}$.

We consider the case of large Schmidt numbers, $\mathrm{Sc}=\nu/D\gg 1$. This condition is always satisfied for Brownian particles. The solution of Eq. (7) can be obtained using an asymptotic analysis (see, e.g., [3,8–10,17,23]). This analysis is based on the separation of scales. In particular, the solution of the Schrödinger equation (7) with a variable mass has different regions where the form of the potential $U_0(r)$, mass $m(r)$ and, therefore, eigenfunctions $\Psi(r)$ are different. The functions Φ and Φ' in these different regions can be matched at their boundaries. Note that the most important part of the solution is localized in small scales (i.e., $r\ll 1$). The results obtained by this asymptotic analysis are presented below. Consider three-dimensional turbulent fluid flow ($d=3$). The solution of Eq. (7) has several characteristic regions. In region I, i.e., for $0\leqslant r\ll l_\nu$ the function $\chi=-\lambda_1\alpha m(r)r$ and $\kappa=20\alpha\sigma/(1+\sigma)$ and $1/m=2(1+X^2)/\mathrm{Pe}$, where $\lambda_1=2(12\sigma-1)/3(1+\sigma)$, and the parameter of compressibility $\sigma=\varepsilon/(1-\varepsilon)$. Using these functions and Eqs. (6)–(8) we get formulas for the potential $U_0(r)$ and the functions $\Psi(r)$ and $\Phi(r)$,

$$U_0\sim 2\alpha\beta_m\left[\zeta(\zeta+1)+\frac{1-\mu^2}{1+X^2}\right],$$
$$\Psi=(1+X^2)^{1/2}S(X),\quad \Phi(r)=\frac{1}{X}(1+X^2)^{\mu/2}S(X), \tag{12}$$

where $X=(\alpha\beta_m\mathrm{Pe})^{1/2}r$, $S(X)=\mathrm{Re}\{A_1P_\zeta^\mu(iX)+A_2Q_\zeta^\mu(iX)\}$ is a real part of the complex function, $P_\zeta^\mu(Z)$ and $Q_\zeta^\mu(Z)$ are the Legendre functions with imaginary argument $Z=iX$, $\beta_m=(1+3\sigma)/3(1+\sigma)$, and

$$\mu=15\sigma/(1+3\sigma), \tag{13}$$

$$\zeta(\zeta+1)=\mu^2-5\mu+2. \tag{14}$$

Condition $\Phi(r=0)=\text{const}$ yields the ratio A_1/A_2. The correlation function has a global maximum at $r=0$ and therefore it satisfies the conditions: $\Phi'(r=0)=0$, and $\Phi''(r=0)<0$, and $\Phi(r=0)>|\Phi(r>0)|$. The function Φ for $X\ll 1$ [i.e., for $0\leqslant r\ll(\alpha\mathrm{Pe})^{-1/2}$] is given by

$$\Phi\sim B_1\left\{1-\frac{\mu}{3}\left[X^2-\frac{(3-\zeta-\mu)(4+\zeta-\mu)}{20}X^4+O(X^6)\right]\right\}, \tag{15}$$

where $B_1\sim A_2$ and we use Eqs. (13) and (14). It follows from Eq. (15) that for the incompressible velocity field $\sigma=0$ (i.e., $\mu=0$) the correlation function $\Phi=B_1=$ constant. The same result follows from the solution of Eq. (2) for $b=0$ (i.e., $\sigma=0$):

$$\Phi=B_1+B_2\int[m(r)/r^2]dr. \tag{16}$$

The integral in Eq. (16) has a singularity at $r=0$, and therefore a unique nonsingular solution for the correlation function $\Phi(r)$ for incompressible velocity field is a trivial solution $\Phi=B_1$ (i.e., $B_2=0$). On the other hand, when $\sigma\neq 0$ there is no singularity in the function $\Phi(r)$.

Now we consider an asymptotic solution of Eq. (7) for $X\gg 1$ [i.e., for $(\alpha\mathrm{Pe})^{-1/2}\ll r\ll l_\nu$]. A form of the solution depends on the parameter $\zeta=-[1\pm(4\mu^2-20\mu+9)^{1/2}]/2$. When $\sigma>1/27$ (i.e., $\mu>1/2$) the parameter $\zeta=-1/2\pm i\zeta_i$ is a complex number, and the correlation function is given by

$$\Phi=AX^{\mu-3/2}\cos(\zeta_i\ln r+\varphi_0), \tag{17}$$

where $\zeta_i=|4\mu^2-20\mu+9|^{1/2}/2$. This solution is valid for $1/27<\sigma<1/7$ (i.e., $1/2<\mu<3/2$). The parameters A and φ_0 are determined by the coefficients A_1 and A_2. When $\sigma<1/27$ (i.e., $\mu<1/2$) the parameter $\zeta=-1/2\pm\zeta_i$ is a real number, and the correlation function is given by

$$\Phi=AX^{-(3/2-\mu-\zeta_i)},$$

where $3/2-\mu-\zeta_i<0$ and for small σ the exponent $3/2-\mu-\zeta_i\sim 10\sigma$.

In region II, i.e., for $l_\nu\leqslant r\ll r_a$ the function $\chi=-\lambda_2/r$, $\kappa=2\sigma q(q-1)(q+2)r^{q-3}/3(1+\sigma)$, and $1/m=2\beta r^{q-1}$, where $\lambda_2=(q-1)B(q,\sigma)/2(1+q\sigma)$, $B(q,\sigma)=2\sigma(q+3)-1$, and $\beta=(1+q\sigma)/3(1+\sigma)$. Using these functions and Eq. (8) we calculate the potential $U_0(r)$,

$$U_0\sim-\frac{1+4c^2}{4mr^2},$$

where $c=\sqrt{M(q,\sigma)}/2(q+3)(1+q\sigma)$, $M(q,\sigma)=b_1B^2+b_2B+b_3$, $b_1(q)=-8q^3+7q^2+36q-36$, $b_2(q)=2(8q^4+26q^3+17q^2-24q-36)$, and $b_3(q)=3(4q^4+12q^3-3q^2-28q-12)$. Note that $M(\sigma=0)=-4q^2(q+3)^2$. The solution of Eq. (7) in this region depends on the parameter of compressibility σ. When σ is in the range: $\sigma_1<\sigma<\sigma_2$, the functions $\Psi(r)$ and $\Phi(r)$ are given by

$$\Psi=A_3r^{1/2}\cos(c\ln r+\varphi_1),\quad \Phi(r)=\Psi/r^{a+1/2}, \tag{18}$$

where $a=[q-\sigma(2q^2+3q-6)]/2(1+q\sigma)$, and $\sigma_i=(B_i+1)/2(q+3)$, and B_i are the roots of the equation $M=0$. When $0<\sigma<\sigma_1$ the functions $\Psi(r)$ and $\Phi(r)$ are given by

$$\Psi=r^{1/2}(A_3r^{|c|}+A_4r^{-|c|}),\quad \Phi(r)=\Psi/r^{a+1/2}. \tag{19}$$

In region III $r_a\ll r\ll 1$ the function $\chi=-\lambda_3/r$, $\kappa=0$, and $1/m=2r^{q-1}/3$, where $\lambda_3=-(q-1)/2$. Using these functions and Eqs. (6)–(8) we get formulas for the potential $U_0(r)$ and the functions $\Psi(r)$ and $\Phi(r)$,

$$U_0\sim\frac{q^2-1}{4mr^2},$$
$$\Psi=r^{1/2}(A_5r^{q/2}+A_6r^{-q/2}),\quad \Phi(r)=A_5+A_6r^{-q}, \tag{20}$$

When $r_a < l_\nu$ (i.e., $\mathrm{Re} < 1/r_a^{3-p}$) region II vanishes and the solution (17) for $0 < r < l_\nu$ borders on the solution (20) for $l_\nu < r < 1$.

In large scales, i.e., $r \gg 1$, functions $F(r)$ and $F_c(r)$ tend to zero and, therefore $1/m(r) \to 2/3$ and $U_0 \to 0$. The correlation function in this range is given by $\Phi(r) = A_7 r^{-1} \exp(-r\sqrt{3|\gamma|/2})$, Parameters A_k, φ_1 and the growth rate of fluctuations γ are determined by matching functions $\Phi(r)$ at the boundaries of these regions. In particular, the growth rate of fluctuations of particles concentration is given by

$$\gamma = \frac{2[c^2 + (q-a)^2]^2}{3q^4(3-p)^2 r_a^{2q}} \ln^2\left(\frac{\mathrm{Re}}{\mathrm{Re}^{(\mathrm{cr})}}\right), \tag{21}$$

where $r_a = (\tau_p/\tau_0)^{1/(p-1)}$, $\mathrm{Re} > \mathrm{Re}^{(\mathrm{cr})}$, and for the critical Reynolds number $\mathrm{Re}^{(\mathrm{cr})}$,

$$\mathrm{Re}^{(\mathrm{cr})} \simeq r_a^{p-3} \exp\left[\frac{3-p}{c}\left(\pi k + \arctan\frac{q-a}{c} + \arctan\frac{3/2 - \mu - a + c_* \zeta_i}{c}\right)\right], \tag{22}$$

where $k = 1,2,3,\ldots$ and $c_* = \tan(\zeta_i \ln l_\nu + \varphi_0)$. Therefore, the fluctuations of particle concentration can be excited without an external source.

The divergent velocity field of inertial particles is the main reason for the self-excitation (exponential growth) of fluctuations of a concentration of small particles in a turbulent fluid flow. Indeed, multiplication of Eq. (1) by n_p and simple manipulations yield

$$\frac{\partial n_p^2}{\partial t} + (\vec{\nabla} \cdot \mathbf{S}) = -n_p^2(\vec{\nabla} \cdot \mathbf{U}) - 2D(\vec{\nabla} n_p)^2, \tag{23}$$

where $\mathbf{S} = n_p^2 \mathbf{U} - D\vec{\nabla} n_p^2$. The latter equation implies that if $\vec{\nabla} \cdot \mathbf{U} < 0$, a perturbation of the equilibrium homogeneous distribution of inertial particles can grow in time, i.e., $(\partial/\partial t)\int n_p^2 d^3 r > 0$. However, the total number of particles is conserved. Averaging Eq. (23) over a volume V_* we obtain

$$\frac{\partial \langle n_p^2 \rangle}{\partial t} \sim -\langle n_p^2 (\vec{\nabla} \cdot \mathbf{U}) \rangle - 2D\langle (\vec{\nabla} n_p)^2 \rangle. \tag{24}$$

Here we used $\int(\vec{\nabla} \cdot \mathbf{S}) dV_* = \int \mathbf{S} \cdot d\mathbf{A} \ll \int n_p^2 (\vec{\nabla} \cdot \mathbf{U}) dV_*$, and $\mathbf{A}$ is a closed surface. Equation (24) implies that the variation of particle concentration during the time interval $\tau_0 = l_0/u_0$, around the value $n_p^{(0)}$ is of the order of $\delta n_p \sim -n_p^{(0)} \tau_0 (\vec{\nabla} \cdot \mathbf{U})$, where u_0 is the characteristic velocity in the energy containing scale l_0. Substituting $n_p = n_p^{(0)} + \delta n_p$ into Eq. (24) yields $\partial \langle n_p^2 \rangle / \partial t \sim 2\tau_0 \langle n_p^2 (\vec{\nabla} \cdot \mathbf{U})^2 \rangle$. Therefore, the growth rate of fluctuations of particles concentration $\gamma \sim 2\tau_0 \langle (\vec{\nabla} \cdot \mathbf{U})^2 \rangle$. This estimate in a good agreement with the analytical results obtained above.

The physics of self-excitation (exponential growth) of fluctuations of particle concentration is as follows. The inertia causes particles inside the turbulent eddy to drift out to the boundary regions between eddies (the regions with maximum pressure of the fluid). Indeed, Eq. (4) shows that particles inertia results in $\vec{\nabla} \cdot \mathbf{U} \propto \tau_p \Delta P/\rho$. On the other hand, for large Péclet numbers $\vec{\nabla} \cdot \mathbf{U} \propto -dn_p/dt$ [see Eq. (1)]. Therefore, $dn_p/dt \propto -\tau_p \Delta P/\rho$. Thus there is accumulation of inertial particles (i.e., $dn_p/dt > 0$) in regions with the maximum pressure of a turbulent fluid (i.e., where $\Delta P < 0$). Similarly, there is an outflow of inertial particles from the regions with the minimum pressure of fluid.

This mechanism acts in a wide range of scales of a turbulent fluid flow. Turbulent diffusion results in relaxation of fluctuations of particle concentration in large scales. However, in small scales where turbulent diffusion is small, the relaxation of fluctuations of particle concentration is very weak. Therefore the fluctuations of particle concentration are localized in the small scales.

This phenomenon is considered for the case when density of fluid is much less than the material density of particles $(\rho \ll \rho_p)$. When $\rho \gg \rho_p$ the results coincide with those obtained for the case $(\rho \ll \rho_p)$ except for the transformation $\tau_p \to \beta_* \tau_p$. For $\rho \gg \rho_p$ the value $dn_p/dt \propto -\beta_* \tau_p \Delta P/\rho$. Thus there is accumulation of inertial particles (i.e., $dn_p/dt > 0$) in regions with the minimum pressure of a turbulent fluid since $\beta_* < 0$.

III. ANOMALOUS SCALING FOR FLUCTUATIONS OF PARTICLES CONCENTRATION

Problems of anomalous scalings for vector (magnetic) and scalar (particles number density or temperature) fields passively advected by a turbulent fluid flow are a subject of an active research in the last years (see, e.g., [10–15,18]). For an incompressible turbulent flow and when $\mathbf{U} = \mathbf{v}$ the anomalous scalings for scalar field can occur only beginning with a fourth-order correlation function (see e.g., [12–14]). In this section we show that the anomalous scaling appears already in the second moment of the number density of particles when the degree of compressibility $\sigma > 1/27$.

We study the case when there is no self-excitation of the fluctuations of the number density of inertial particles, i.e., the case when $\mathrm{Re} < \mathrm{Re}^{(\mathrm{cr})}$. Consider fluctuations of inertial particle concentration in the presence of a source $I(r)$ and we study a zero mode for Eq. (2), i.e., the mode with $\gamma = 0$. Substituting Eq. (6) in Eq. (2) yields an equation for the unknown function $\psi(t,r)$,

$$\frac{\partial \psi}{\partial t} = \frac{1}{m} \frac{\partial^2 \psi}{\partial r^2} - U_0(r)\psi + \tilde{f}(r), \tag{25}$$

where $\tilde{f}(r) = rI(r)\exp[\int_0^r \chi(x) dx]$. Determine the stationary solution of Eq. (25). The external source in these scales is chosen as follows: $I(r) = I_0(1 - r^s)$, where $s > 0$, and for $r > 1$, $I(r) = 0$. The general solution of Eq. (25) reads

$$\psi(r) = A\Psi_1 + B\Psi_2 + \int_0^\infty G(r,\xi)\tilde{f}(\xi) d\xi, \tag{26}$$

where Ψ_1 and Ψ_2 are solutions of Eq. (25) with $I = 0$, and Green function $G(r,\xi)$ is given by

$$G(r,\xi)=m(\xi)H(r-\xi)\frac{\Psi_1(r)\Psi_2(\xi)-\Psi_2(r)\Psi_1(\xi)}{\Psi_1(\xi)\Psi_2'(\xi)-\Psi_2(\xi)\Psi_1'(\xi)},$$

and $H(y)$ is a Heaviside function. Equation (26) yields the formula for the second moment $\Phi(r)$ in different regions. When $l_\nu \leqslant r < r_a$

$$\Phi=r^{-a}(A_3 r^{|c|}+A_4 r^{-|c|})-\frac{I_0}{2\beta_m[(3-q+a)^2-c^2]}r^{3-q}$$

[for $1/27<\sigma<\min(\sigma_1,1/7)$], and

$$\Phi=A_3 r^{-a}\cos(c\ \ln r+\varphi_1)-\frac{I_0}{2\beta_m[(3-q+a)^2+c^2]}r^{3-q}$$

(for $\sigma_1<\sigma<1/7$). When $r_a<r\ll 1$

$$\Phi(r)=A_5+A_6 r^{-q}-\frac{I_0}{2(3-q)}r^{3-q},$$

and when $r\gg 1$ the function $\Phi(r)=A_7/r$. Matching functions $\Phi(r)$ and $\Phi'(r)$ at the boundaries of these regions yields the constant A_k. The term $\propto r^{3-q}$ in these equations corresponds to a normal scaling for the second moment of inertial particles concentration, whereas the term $\propto r^{-q}$ in the range $r_a<r\ll 1$ corresponds to the anomalous scaling. When $\sigma_1<\sigma<1/7$ the anomalous scaling in the range $l_\nu \leqslant r<r_a$ is complex ($\propto r^{-a\pm i|c|}$).

IV. EFFECT OF CHEMICAL REACTIONS (OR PHASE TRANSITIONS) ON SELF-EXCITATION AND ANOMALOUS SCALING OF FLUCTUATIONS OF PARTICLES NUMBER DENSITY

Now we study fluctuations of the number density of small particles in a turbulent fluid flow with chemical reactions (or heterogeneous phase transitions, e.g., evaporation or condensation). The source of particles (or droplets) is I_c. Consider a homogeneous equilibrium with $I_c=0$. Now we study deviation from this equilibrium. Linearizing Eq. (1) with the source I_c for the number density of small particles in the vicinity of the equilibrium we obtain an equation for small perturbations,

$$\frac{\partial n_p}{\partial t}+\vec{\nabla}\cdot(n_p\mathbf{U})=D\Delta n_p-(\tau_0/\tau_c)n_p \tag{27}$$

(see [24]), where $\tau_0/\tau_c=-\partial I_c/\partial n_p$, and τ_c is the characteristic time of chemical reaction (or phase transition). Equation for the second moment of the number density of particles coincides with Eq. (2) except for the change $\langle \tau b(\mathbf{x})b(\mathbf{y})\rangle \to \langle \tau b(\mathbf{x})b(\mathbf{y})\rangle-\tau_0/\tau_c$. In this case all equations (6)–(11) are not changed except for the change $U_0 \to U_0+\tau_0/\tau_c$. Therefore in this case we can use the same analysis that is performed in Secs. II and III. Indeed, consider a solution of the equation for the second moment of the number density of particles. In region I, $0<r<l_\nu$, a nonsingular solution for the correlation function exists only for a compressible fluid flow when $\sigma>1/27$ and $\mathrm{Re}>\mathrm{Re}^*$, where $\mathrm{Re}^*=(\tau_0/\tau_c)^{(3-p)/(3-q)}$ [see the comments after Eqs. (15) and (16)]. The condition $\mathrm{Re}>\mathrm{Re}^*$ means that the solution is independent of the chemical reactions (or phase transitions). This solution coincides with that given by Eq. (12). On the other hand, for an incompressible fluid flow ($\sigma=0$) with chemical reactions (or phase transitions) and without an external pumping only a solution with $(\partial\Phi/\partial r)_{r\to 0}>0$ exists for small but finite molecular diffusion. This solution cannot be a correlation function.

Consider a solution for the second moment of the number density of particles in the region II, $l_\nu<r<r_c<r_a$, where the scale $r_c=(\tau_c/\tau_0)^{1/(3-q)}$ is determined by the condition $1/(mr^2)\sim\tau_0/\tau_c$, i.e., when the effect of chemical reactions (or phase transitions) is essential. When $1/mr^2\ll\tau_0/\tau_c$, the function Ψ is determined by an equation $r^{q-1}\Psi''-(\tau_0/\tau_c\beta)\Psi=0$. The solution of this equation is given by

$$\Psi=A_3 r^{1/2}K_{1/(3-q)}\left(\frac{2}{3-q}\sqrt{\frac{\tau_0}{\tau_c\beta}}r^{(3-q)/2}\right)$$
$$\sim r^{(q-1)/4}\exp\left(-\frac{2}{3-q}\sqrt{\frac{\tau_0}{\tau_c\beta}}r^{(3-q)/2}\right),\quad \Phi=\Psi/r^{a+1/2}, \tag{28}$$

where $K_\nu(y)$ is the modified Bessel function of the second type. Since $\sqrt{\tau_0/\tau_c\beta}r^{(3-q)/2}\ll 1$ the effect of chemical reactions (or phase transitions) causes strong localization of the solution given by Eq. (28) in the scales $r\gg r_c$. When $l_\nu<r\ll r_c$ the effect of chemical reactions (or phase transitions) is negligible and the correlation function Φ is given by Eq. (18) which is valid for $\max(\sigma_1,1/27)<\sigma<\sigma_2$ (see Sec. II). The latter solution determines the self-excitation of fluctuations of particles number density when $\mathrm{Re}>\mathrm{Re}^{(\mathrm{cr})}$, and the complex anomalous scaling when $\mathrm{Re}<\mathrm{Re}^{(\mathrm{cr})}$ (see Secs. II and III). Solutions (18) and (28) are matched in the vicinity $r\sim r_c$. This means that in the case of chemical reactions (or phase transitions) there is a possibility for both, the self-excitation of fluctuations of the particles number density when $\mathrm{Re}>\mathrm{Re}^{(\mathrm{cr})}$, and for the complex anomalous scaling when $\mathrm{Re}<\mathrm{Re}^{(\mathrm{cr})}$ and if there is an external pumping.

On the other hand, solution (28) with its first derivative cannot be matched with that given by Eq. (19). The latter solution is valid for $1/27<\sigma<\sigma_1$ and determines the real anomalous scaling. Consider the case $r_a<r_c\ll 1$. The effect of chemical reactions (or phase transitions) is essential for $r\gg r_c$ and the solution for the second-order correlation function Φ of the particles number density is given by Eq. (28) with $\beta=2/3$. When $r_a<r\ll r_c$ the correlation function Φ is determined by Eq. (20). These two solutions with their first derivatives cannot be matched. Solution (20) determines also the real anomalous scaling. This means that in the case of chemical reactions (or phase transitions) the real anomalous scaling does not exist.

Remarkably, chemical reactions (or phase transitions) do not affect the threshold $\mathrm{Re}^{(\mathrm{cr})}$ for self-excitation of fluctuations of the particles number density and the complex anomalous scaling of the fluctuations. Chemical reactions (or phase transitions) in the case $r_a>r_c$ only cause strong localization of solution for the second-order correlation function

Φ in the vicinity of $r\sim r_c$. Note also that when $\tau_c \leqslant \tau_0$ (i.e., $r_c > 1$) the effect of chemical reactions (or phase transitions) is negligible.

The physics of the effect of strong localization of fluctuations due to chemical reactions (or phase transitions) can be illucidated using an example of the simple irreversible chemical reaction $A\to C$. During transport of an admixture A by a turbulent fluid flow the number density of the admixture A decreases due to the chemical reaction $A\to C$ with a very short time $\tau_c \ll \tau_0$. Thus turbulent diffusion does not contribute to the mass flux of a reagent A. Turbulent mixing is effective only for the product C of the reaction. The situation becomes more complicated for a multicomponent chemical reaction with the inverse reaction. The effect of depletion of the turbulent diffusion is similar to that for the reaction $A \to C$. The total chemical relaxation time τ_c is determined by an equation $\tau_c^{-1} = \Sigma_{j=1}^{k} \tilde{\tau}_j^{-1}$, where $\tilde{\tau}_j$ is the relaxation time of the j component. The components with time $\tau_i \gg \tau_c$ have the turbulent diffusion coefficient $\sim D_T$, whereas turbulent diffusion coefficients for the components with $\tau_i \sim \tau_c$ are strongly reduced (for details see [24]). This effect of strong depletion of turbulent diffusion can be interpreted as strong localization of the fluctuations of particle number density.

V. TURBULENT FLUX OF PARTICLES

Now we study the large-scale dynamics of inertial particles. The evolution of the number density $n_p(t,\mathbf{r})$ of small particles in a turbulent flow is determined by the equation

$$\frac{\partial n_p}{\partial t} + \vec{\nabla}\cdot(n_p \mathbf{U}) = -\vec{\boldsymbol{\nabla}}\cdot \mathbf{J}_M, \tag{29}$$

where the flux of particles $\mathbf{J}_M$ is given by $\mathbf{J}_M = -D[\vec{\boldsymbol{\nabla}} n_p + k_t \vec{\boldsymbol{\nabla}} T_f / T_f + k_p \vec{\boldsymbol{\nabla}} P_f / P_f]$. The first term in the formula for the flux of particles describes molecular diffusion, while the second term accounts for the flux of particles caused by the temperature gradient $\vec{\boldsymbol{\nabla}} T_f$ (molecular thermal diffusion for gases or thermophoresis for particles, see, e.g., [25]), and the third term determines the flux of particles caused by the pressure gradient $\vec{\nabla} P_f$ (molecular barodiffusion). Here $k_t \propto n_p$ is the thermal diffusion ratio, Dk_t is the coefficient of thermal diffusion, $k_p \propto n_p$ is the barodiffusion ratio, Dk_p is the coefficient of barodiffusion, and T_f and P_f are the temperature and pressure of surrounding fluid, respectively.

We consider here the case of large Reynolds and Péclet numbers and do not take into account the effect of particles upon the carrying fluid flow. The solution of the equation of motion for small particles with $\rho_p \gg \rho$ yields

$$\mathbf{U} = \mathbf{v}(t,\mathbf{Y}(t)) - \tau_p[\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\vec{\nabla})\mathbf{v}] + \mathrm{O}(\tau_p^2), \tag{30}$$

(see, e.g., [20]), where $\mathbf{v}$ is the velocity of the surrounding fluid, $\mathbf{Y}(t)$ is the position of the particle, ρ_p is the material density of particles, and ρ is the density of the fluid.

In this study we consider a low-Mach-number compressible turbulent flow $\vec{\nabla}\cdot\mathbf{v} \neq 0$. The velocity field of particles is also compressible, i.e., $\vec{\nabla}\cdot\mathbf{U} \neq 0$. Equation (30) for the velocity of particles and Navier-Stokes equation for the fluid for large Reynolds numbers yields

$$\begin{aligned}\vec{\nabla}\cdot\mathbf{U} &= \vec{\nabla}\cdot\mathbf{v} - \tau_p \vec{\nabla}\cdot(d\mathbf{v}/dt) + \mathrm{O}(\tau_p^2)\\ &= \vec{\nabla}\cdot\mathbf{v} + \tau_p \vec{\nabla}\cdot(\vec{\nabla} P_f/\rho) + \mathrm{O}(\tau_p^2).\end{aligned} \tag{31}$$

We study the large-scale dynamics of small inertial particles and average Eq. (29) over an ensemble of random velocity fluctuations. For this purpose we use the method of path integrals. The equation for the mean field $N = \langle n_p \rangle$ is

$$\frac{\partial N}{\partial t} + \vec{\boldsymbol{\nabla}}\cdot[N\mathbf{V}_{\mathrm{eff}} - \hat{D}\vec{\boldsymbol{\nabla}}_m N] = 0 \tag{32}$$

(see Appendix A), where $\hat{D} \equiv D_{pm}$ and $\mathbf{V}_{\mathrm{eff}} = \mathbf{V}_p - \langle \tau b \mathbf{u} \rangle$. Equation (32) was derived for $\mathrm{Pe} \gg 1$. It can be shown that for $\mathrm{Pe} \ll 1$ and arbitrary velocity field the equation for the mean field coincides with Eq. (32).

Now we calculate the velocity $\mathbf{V}_{\mathrm{eff}}$. Using the equation of state $P_f = \kappa_B T_f \rho / m_\mu$ and Eq. (31) we obtain $\langle \tau \mathbf{u} b \rangle \approx \langle \tau(\vec{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{u}})\tilde{\mathbf{u}} \rangle + (\tau_p v_T^2 / T_*)\langle \tau \tilde{\mathbf{u}} \Delta\theta \rangle$, where $v_T^2 = \kappa_B T_* / m_\mu$, m_μ is the mass of molecules of surrounding fluid and $T_f(t,\mathbf{r})$ is the temperature field with a characteristic value T_*, θ are fluctuations of temperature, κ_B is Boltzmann constant. We neglect here the second moments $\sim\langle \tilde{\mathbf{u}}\tilde{\rho} \rangle$, since the mean turbulent mass flux of the surrounding fluid vanishes in a finite domain surrounded by solid boundaries. Here $\tilde{\rho}$ and $\tilde{\mathbf{u}}$ are fluctuations of the density and velocity of the fluid. On the other hand, the mean turbulent heat flux $\langle \tilde{\mathbf{u}}(\mathbf{x})\theta(\mathbf{x}) \rangle$ is nonzero in the presence of an external mean temperature gradient, i.e., $\langle \tilde{\mathbf{u}}(\mathbf{x})\theta(\mathbf{x}) \rangle = -\chi_T \vec{\boldsymbol{\nabla}} T$, where the total temperature is $T_f = T + \theta, T = \langle T_f \rangle$ is the mean temperature field, $\chi_T \sim u_0 l_0 / 3$ is the coefficient of turbulent thermal diffusivity. Therefore, the effective velocity is given by

$$\mathbf{V}_{\mathrm{eff}} = \mathbf{V}_p - \langle \tau(\vec{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{u}})\tilde{\mathbf{u}} \rangle - \frac{l_0 u_0}{\mathrm{Pe}}\left(\frac{m_p}{m_\mu}\right)\ln(\mathrm{Re}_*)\vec{\boldsymbol{\nabla}} T,$$

where $\mathrm{Re}_* = \mathrm{Re}\, F_0^{1/2}, \mathrm{Re} = l_0 u_0 / \max(\nu_0, \chi_0)$ is the Reynolds number, χ_0 is the coefficient of molecular thermal conductivity, and $F_0(\mathbf{r}) = \langle \tilde{\mathbf{u}}^2(\mathbf{r}) \rangle / u_0^2$. We use here an identity

$$\frac{\tau_p v_T^2}{l_0 u_0} = \frac{1}{\mathrm{Pe}}\left(\frac{m_p}{m_\mu}\right),$$

and $\mathrm{Pe} = u_0 l_0 / D_*$ is the Péclet number and the molecular diffusion coefficient $D_* = \kappa T_* / (6\pi a_* \rho \nu)$.

Equation (32) with this effective velocity $\mathbf{V}_{\mathrm{eff}}$ can be rewritten in the form

$$\frac{\partial N}{\partial t} + \vec{\boldsymbol{\nabla}}\cdot(N\mathbf{V}_p) = -\vec{\boldsymbol{\nabla}}\cdot(\mathbf{J}_T + \mathbf{J}_M), \tag{33}$$

where

$$\mathbf{J}_T = -D_T\left[\frac{k_T}{T}\vec{\boldsymbol{\nabla}} T - \frac{k_P}{P}\vec{\boldsymbol{\nabla}} P + F_0 \vec{\boldsymbol{\nabla}} N\right], \tag{34}$$

$$k_T = N[F_0 + T(\eta_0 + \sigma_0 f)], \tag{35}$$

$$\eta_0 = \frac{3}{\mathrm{Pe}}\left(\frac{m_p}{m_\mu}\right)\left(\frac{1}{T_*}\right)\ln \mathrm{Re}, \tag{36}$$

where $\sigma_0 = \eta_0/(2\ln \mathrm{Re}), f=\ln F_0$, and $D_T = u_0 l_0/3$ is the coefficient of turbulent diffusion, k_T can be interpreted as the turbulent thermal diffusion ratio, and $D_T k_T$ is the coefficient of turbulent thermal diffusion; $k_P = F_0 N$ can be interpreted as the turbulent barodiffusion ratio and $D_T k_P$ is the coefficient of turbulent barodiffusion. Note that for $\mathrm{Re} \gg 1$ and $\mathrm{Pe} \gg 1$ both turbulent diffusion coefficients are much larger than the corresponding molecular coefficients (i.e., $D_T \gg D$, and $D_T k_T \gg D k_t$), and $D_T k_P \gg D k_p$. Using Eq. (30) the particle mean velocity can be written in the form

$$(\mathbf{V}_p)_i = \mathbf{V}_i - \tau_p \frac{\partial \mathbf{V}_i}{\partial t} - \tau_p \frac{\partial}{\partial x_j}\langle u_i u_j\rangle + \tau_p \langle u_j b\rangle. \tag{37}$$

Taking into account Eq. (37) the turbulent flux of particles $\mathbf{J}_T^*$ in isotropic turbulence is given by

$$\mathbf{J}_T^* = \mathbf{J}_T - \frac{\tau_p}{3}\vec{\boldsymbol{\nabla}}\langle \mathbf{u}^2\rangle, \tag{38}$$

where $\mathbf{J}_T$ is determined by Eq. (34) and

$$k_T = N\left[F_0\left(1+\frac{\tau_p}{\tau_0}\right) + T(\eta_0 + \sigma_0 f)\right],$$

$$k_P = N F_0\left(1+\frac{\tau_p}{\tau_0}\right).$$

The second term in Eq. (38) describes the effect of turbophoresis (see [26,27]).

Compressibility of the background fluid is important when the size of particles smaller than one micrometer (or for the gaseous admixture). In this case the effect of particle inertia is very small and the main contribution to the effect of the turbulent thermal diffusion is due to the compressibility of the background fluid. On the other hand, when the size of particles is larger than 5–10 μm the effect of particles inertia is very important and the contribution to the effect of the turbulent thermal diffusion caused by particles inertia is much larger than that due to compressibility of the background fluid [i.e., $T(\eta_0 + \sigma_0 f) \ll F_0$, see Eq. (35)]. Certainly, the compressibility ($\boldsymbol{\nabla}\cdot\mathbf{v} \neq 0$) of the background fluid cannot be ignored completely since otherwise we cannot satisfy the continuity equation and the equation of state simultaneously in the presence of a nonzero mean temperature gradient.

VI. LARGE-SCALE INSTABILITY

Let us study the large-scale dynamics. The equilibrium solution of Eq. (33) can be unstable. The mechanism of the instability for $\rho_p \gg \rho$ is as follows. The inertia causes particles inside the turbulent eddy to drift out to the boundary regions between eddies (the regions with decreased velocity of the turbulent fluid flow and maximum of pressure of the surrounding fluid). Thus, inertial particles are accumulated in regions with the maximum pressure of the turbulent fluid. Indeed, the inertia effect results in $\vec{\nabla}\cdot\mathbf{U} \propto \tau_p \Delta P_f \neq 0$. On the other hand, for large Peclet numbers $\vec{\nabla}\cdot\mathbf{U} \propto -dn_p/dt$. The latter implies that in regions with the maximum pressure of turbulent fluid (i.e., where $\Delta P_f < 0$) there is an accumulation of inertial particles (i.e., $dn_p/dt > 0$). Similarly, there is an outflow of inertial particles from regions with the minimum pressure of fluid. In a homogeneous and isotropic turbulence without large-scale external gradients of temperature a drift from regions with increased (decreased) concentration of inertial particles by a turbulent flow of fluid is equiprobable in all directions. Therefore the pressure (temperature) of the surrounding fluid is not correlated with the turbulent velocity field and there exists only a turbulent diffusion flux of inertial particles.

The situation is drastically changed when there is a large-scale inhomogeneity of the temperature of the turbulent flow. In this case the mean heat flux $\langle \tilde{\mathbf{u}}\theta\rangle \neq 0$. Therefore fluctuations of both temperature and velocity of fluid are correlated. Fluctuations of temperature cause fluctuations of the pressure of the fluid. The pressure fluctuations result in fluctuations of the concentration of inertial particles. Indeed, an increase (decrease) of the pressure of the surrounding fluid is accompanied by accumulation (outflow) of the particles. Therefore, the direction of the mean flux of particles coincides with that of the heat flux, i.e., $\langle \tilde{\mathbf{u}} n_p\rangle \propto \langle \tilde{\mathbf{u}}\theta\rangle \propto -\vec{\boldsymbol{\nabla}} T$. The mean flux of the inertial particles is directed to the minimum of the mean temperature and the inertial particles are accumulated in this region.

The evolution of the mean field N is determined by Eq. (33). Substitution

$$N(t,\mathbf{r}) = N_* \Psi_0(Z)\exp(\gamma_0 t)\exp\left[-\frac{1}{2}\int \chi_0(Z)dZ + i\mathbf{k}\cdot\mathbf{r}_\perp\right] + N_0(\mathbf{r})$$

reduces Eq. (33) to the eigenvalue problem for the Schrödinger equation,

$$\frac{1}{m_0}\Psi_0''(Z) + [W_0 - U_0(Z)]\Psi_0(Z) = 0, \tag{39}$$

where $W_0 = -\gamma_0$, $A' = dA/dZ$, and the potential U_0 is given by

$$U_0 = \frac{1}{m_0}\left(\frac{\chi_0^2}{4} + \frac{\chi_0'}{2} + \kappa_0\right),$$

and

$$\chi_0 = f' + \frac{T'}{T} - \frac{P'}{P} + \frac{1}{F_0}(\eta_0 + \sigma_0 f)T',$$

$$\kappa_0 = k^2 - \left(\frac{T'}{T}\right)' + \left(\frac{P'}{P}\right)' + \frac{f'P'}{P} - \frac{1}{F_0}(\eta_0 + \sigma_0 f)T'' - \frac{f'T'}{T}\left(1 + \frac{\sigma_0 T}{F_0}\right).$$

Here $m_0 = \exp[-f(Z)]$, the axis Z is directed along the mean temperature gradient, and the wave vector $\mathbf{k}$ is normal to the axis Z. In deriving Eq. (39) we take into account that for an isotropic turbulence $\langle u_m(\mathbf{x})u_n(\mathbf{x})\rangle = u_0^2 \exp(f)\delta_{mn}/3$. Equilibrium distribution of the mean number density $N_0(\mathbf{r})$ is determined by equation $\hat{D}\vec{\boldsymbol{\nabla}}_m N_0 = \mathbf{V}_{\mathrm{eff}} N_0$. Equation (39) is written in the dimensionless form, the coordinate is measured in units Λ_T, time t is measured in units Λ_T^2/D_T, the wave number k is measured in units Λ_T^{-1}, the temperature T is measured in units of temperature difference δT in the scale Λ_T, and concentration N is measured in units N_*.

Now we use a quantum mechanical analogy for the analysis of the large-scale pattern formation in the concentration field N of the inertial particles. The instability can be excited ($\gamma_0>0$) if there is a region of a potential well where $U_0<0$. The positive value of W_0 corresponds to the turbulent diffusion, whereas a negative value of W_0 results in the excitation of the instability. Consider the case $P'/P \ll T'/T$. The potential U_0 can be rewritten as

$$U_0=\frac{1}{4m_0}\Bigg[\left(f'-\frac{T'}{T}\right)^2+\left(\frac{T'}{T}-\frac{\sigma_0 Tf'}{F_0}\right)^2+\left[\frac{T'}{T}+\frac{1}{F_0}(\eta_0+\sigma_0 f)T'\right]^2+4k^2+2f''-2\frac{T''}{T}-\frac{2}{F_0}(\eta_0+\sigma_0 f)T''-\left(\frac{\sigma_0 Tf'}{F_0}\right)^2\Bigg]. \tag{40}$$

The potential U_0 can be negative if

$$2f''-2\frac{T''}{T}-\frac{2}{F_0}(\eta_0+\sigma_0 f)T''-\left(\frac{\sigma_0 Tf'}{F_0}\right)^2<0. \tag{41}$$

In order to estimate the first energy level W_0 we use a modified variational method (e.g., a modified Rayleigh-Ritz method). The modification of the regular variational method is required since Eq. (39) can be regarded as the Schrödinger equation with a variable mass $m_0(Z)$. Now we rewrite Eq. (39) in the form

$$\hat{H}\Psi_0=W_0\Psi_0, \quad \hat{H}=U_0-\frac{1}{m_0}\frac{d^2}{dZ^2}. \tag{42}$$

The modified variational method employs an inequality

$$W_0\leq I, \quad I=\int m_0\Psi^*\hat{H}\Psi\, dZ, \tag{43}$$

where Ψ is an arbitrary function that satisfies a normalization condition

$$\int m_0\Psi^*\Psi\, dZ=1. \tag{44}$$

The inequality (43) can be proved if one uses the expansion $\Psi=\Sigma_{p=0}^{\infty}a_p\Psi_0^{(p)}$, where $\Sigma_{p=0}^{\infty}|a_p^2|=1$ and $\int m_0(\Psi_0^{(p)})^*\Psi_0^{(k)}dZ=\delta_{pk}$. The eigenfunctions $\Psi_0^{(p)}$ satisfy the equation $\hat{H}\Psi_0^{(p)}=W_p\Psi_0^{(p)}$.

We chose the trial function Ψ in the form

$$\Psi=A_0\exp[-\alpha_0(Z-Z_0)^2/2], \quad A_0=\left(\frac{\alpha_0+b_0}{\pi}\right)^{1/4}\exp\left(\frac{\alpha_0 b_0 Z_0^2}{2(\alpha_0+b_0)}\right), \tag{45}$$

where the unknown parameters α_0 and Z_0 can be found from the condition of the minimum of the function $I(\alpha_0,Z_0)$ [see Eq. (43)]. Here we use the following spatial distributions of $f(Z)$ and $T(Z)$:

$$f(Z)=-b_0Z^2\exp(-\beta_0Z^2), \tag{46}$$

$$T(Z)=(T_*+Z^2+aZ)\exp(-\epsilon_0Z^2), \tag{47}$$

where $\beta_0\ll 1$ and $\epsilon_0\ll 1$. These distributions satisfy the necessary condition (41) for the excitation of the instability. We consider a case $T_*^{-1}\ll b_0$.

Substituting Eqs. (45) and (47) into Eq. (43) yields

$$I=-\eta_0+\frac{1}{2\alpha_0^{3/2}}\{\alpha_0^2(\alpha_0-b_0)^{1/2}+b_0(\alpha_0+b_0)^{1/2}[b_0+2\alpha_0(b_0Z_0^2-1)]\}\times\exp\left(-\frac{\alpha_0 b_0 Z_0^2}{\alpha_0-b_0}\right)+\eta_0^2\frac{(\alpha_0-b_0)^{1/2}}{4(\alpha_0-2b_0)^{5/2}}[2(\alpha_0-2b_0)+(2\alpha_0Z_0+a(\alpha_0-2b_0))^2]\exp\left(\frac{\alpha_0^2 b_0 Z_0^2}{(\alpha_0-b_0)(\alpha_0-2b_0)}\right). \tag{48}$$

Here we consider the case of $k\ll 1$. This implies long-wave perturbations in the horizontal plane. Thus, the modified Rayleigh-Ritz method allows us to estimate the growth rate of the instability. For example, when $b_0\ll\eta_0$ (i.e., the inhomogeneity of turbulence is not strong), the growth rate of the instability in the dimensional form is given by

$$\gamma\geq\frac{3b_0D_T}{2\Lambda_T^2}. \tag{49}$$

Thus, it is shown here that the equilibrium distribution of the number density of particles is unstable. The instability results in the formation of an inhomogeneous distribution of the number density of particles. The exponential growth during the linear stage of the instability can be damped by the nonlinear effects (e.g., hydrodynamic interaction between particles and a turbulent fluid flow, a change of temperature distribution in the vicinity of the temperature inversion layer). The obtained estimate of the growth rate of the instability is in agreement with the numerical solution of Eq. (39).

VII. CONCLUSIONS

Fluctuations of the number density of inertial particles in a turbulent fluid flow are investigated. It is shown that the anomalous scaling appears already in the second moment of the number density of inertial particles when the degree of compressibility of the particles velocity $\sigma>1/27$. It is demonstrated that the inertia of particles in a homogeneous and isotropic turbulent fluid flow causes a self-excitation (exponential growth) of fluctuations of particle concentration. The growth rates of the higher moments of particle concentration is larger than those of the lower moments, i.e., particles spatial distribution is intermittent. This process can be damped by the nonlinear effects (e.g., two-way coupling between fluctuations of particles concentration and turbulent fluid flow). Note that when the particles velocity field is divergence free, i.e., $\vec{\nabla}\cdot\mathbf{U}=0$, all of the moments of the concentration field do not grow and there is no intermittency without an external source of fluctuations of particle concentration. When the inertia effect is negligible (e.g., for small size of particles or gaseous admixture) but the fluid velocity field is divergent, i.e., $\vec{\nabla}\cdot\mathbf{u}\neq 0$, the moments of the concentration field grow and there is intermittency without an external source of fluctuations of particle concentration. In this case Eqs. (21) and (22) with $r_a=1$ determine the growth rate of fluctuations of particle concentration and the critical Reynolds number, respectively. The δ-correlated in time random process is used to describe a turbulent velocity field. However, the results remain valid also for the velocity field with a finite correlation time, if all moments of the number density of the particles vary slowly in comparison with the correlation time of the turbulent velocity field.

The analyzed effect of self-excitation (exponential growth) of fluctuations of particles concentration is important in turbulent fluid flows of a different nature with inertial particles or droplets (e.g., in atmospheric turbulence, combustion, and in laboratory turbulence). In particular, this effect causes formation of inhomogeneities in spatial distribution of fuel droplets in internal combustion engines. The self-excitation of fluctuations of particle concentration is observed in atmospheric turbulence, e.g., this effect causes formation of small-scale inhomogeneities in droplet clouds [5,6]. Small-scale inhomogeneities in the spatial distribution of inertial particles were observed also in laboratory [4].

Large-scale dynamics of inertial particles advected by a low-Mach-number compressible turbulent fluid flow with a nonzero mean temperature gradient is studied as well. The equation for the turbulent flux of particles in a low-Mach-number compressible fluid flow is derived. A modified Rayleigh-Ritz variational method is used for the analysis of the large-scale instability that results in formation of large-scale inhomogeneous structures in the spatial distribution of inertial particles. Note that the large-scale instability can be also interpreted as an inverse cascade of the passive scalar (e.g., particles, number density).

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APPENDIX: DERIVATION OF EQUATIONS FOR THE MEAN FIELD AND THE SECOND-ORDER CORRELATION FUNCTION FOR THE NUMBER DENSITY OF PARTICLES

We study the fluctuations and large-scale dynamics of small inertial particles and average Eq. (1) over an ensemble of random velocity fluctuations. For this purpose we use the method of path integrals whereby the solution of Eq. (1) is reduced to an analysis of the evolution of the concentration field $n_p(t,\mathbf{r})$ along the Wiener path $\vec{\xi}$:

$$\vec{\xi}(t,t_0)=\mathbf{x}-\int_0^{t-t_0}\mathbf{U}[t_s,\vec{\xi}(t,t_s)]ds+(2D)^{1/2}\mathbf{w}(t-t_0), \tag{A1}$$

where $\mathbf{U}=\mathbf{v}_p$, $t_s=t-s$, and $\mathbf{w}(t)$ is a Wiener process. Equation (A1) describes a set of random trajectories that pass through the point $\mathbf{x}$ at time t. The solution of Eq. (1) with the initial condition $n_p(t=t_0,\mathbf{x})=n_0(\mathbf{x})$ is given by the Feynman-Kac formula

$$n_p(t,\mathbf{x})=M\{G(t,t_0)n_0[\vec{\xi}(t,t_0)]\} \tag{A2}$$

(see, e.g., [28]), where

$$G(t,t_0)=\exp\left\{-\int_{t_0}^{t}b_*[\sigma,\vec{\xi}(t,\sigma)]d\sigma\right\}, \tag{A3}$$

$b_*\equiv\vec{\nabla}\cdot\mathbf{U}$, and $M\{\cdot\}$ denotes the mathematical expectation over the Wiener paths.

Now let us derive the equation for the mean field $N=\langle n_p\rangle$ and for the second-order correlation function $\Phi(t,\mathbf{x},\mathbf{y})=\langle\Theta(t,\mathbf{x})\Theta(t,\mathbf{y})\rangle$ for the number density of particles using Eq. (1), where $\Theta=n_p-N$. The procedure of derivation is outlined in the following:

(i) If the total field n_p is specified at instant t, then we can determine the total field $n_p(t+\Delta t)$ at near instant $t+\Delta t$ by means of substitutions $t\to t+\Delta t$ and $t_0\to t$ in Eq. (A2). The result is given by

$$n_p(t+\Delta t,\mathbf{x})=M\{G(t+\Delta t,t)n_p[t,\vec{\xi}(t+\Delta t,t)]\}, \tag{A4}$$

where

$$G(t+\Delta t,t)=\exp\left[-\int_t^{t+\Delta t}b_*(\sigma,\vec{\xi}_\sigma)d\sigma\right],$$

$$\vec{\xi}(t+\Delta t,t)\equiv\vec{\xi}_{\Delta t}=\mathbf{x}-\int_0^{\Delta t}\mathbf{U}(t_\sigma,\vec{\xi}_\sigma)d\sigma+(2D)^{1/2}\mathbf{w}(\Delta t),$$

$t_\sigma=t+\Delta t-\sigma$, and $\vec{\xi}(t_2,t_1)\equiv\vec{\xi}_{t_2-t_1}$, i. e., $\vec{\xi}_\sigma=\vec{\xi}(t+\Delta t,t_\sigma)$.

(ii) Expansion of the functions $n_p(t,\vec{\xi}_{\Delta t})$ and the velocity $U_m(t_\sigma,\vec{\xi}_\sigma)$ in Taylor series in the vicinity of the point $\mathbf{x}$ allows us to express the field $n_p(t,\vec{\xi}_{\Delta t})$ in terms of the field $n_p(t,\mathbf{x})$. Indeed, we expand function $n_p(t,\vec{\xi}_{\Delta t})$ [Eq. (A4)] in Taylor series in the vicinity of the point $\mathbf{x}$:

$$n_p(t,\vec{\xi}_{\Delta t})\simeq n_p(t,\mathbf{x})+\frac{\partial n_p}{\partial x_m}(\vec{\xi}_{\Delta t}-\mathbf{x})_m+\frac{1}{2}\frac{\partial^2 n_p}{\partial x_m\partial x_s}(\vec{\xi}_{\Delta t}-\mathbf{x})_m(\vec{\xi}_{\Delta t}-\mathbf{x})_s+\cdots. \tag{A5}$$

Using equation for the Wiener path we obtain

$$[\vec{\xi}(t_2,t_1)-\mathbf{x}]_m=-\int_0^{t_2-t_1}U_m(t_s,\vec{\xi}_s)ds+(2D)^{1/2}\mathbf{w}_m(t_2-t_1), \tag{A6}$$

where $\vec{\xi}(t_2,t_2-s)\equiv\vec{\xi}_s$. Expanding the velocity $U_m(t_s,\vec{\xi}_s)$ in Taylor series in the vicinity of point $\mathbf{x}$ and using Eq. (A6) yields

$$U_m(t_s,\vec{\xi}_s)\simeq U_m(t_s,\mathbf{x})-U_l\frac{\partial U_m}{\partial x_l}s+(2D)^{1/2}\frac{\partial U_m}{\partial x_l}\mathbf{w}_l(s)+\cdots. \tag{A7}$$

Substituting Eq. (A7) into Eq. (A6) and calculating the integrals in Eq. (A6) accurate up to the terms $\sim O[(t_2-t_1)^2]$ yields

$$[\vec{\xi}(t_2,t_1)-\mathbf{x}]_m \simeq -(t_2-t_1)U_m+\tfrac{1}{2}(t_2-t_1)^2U_l\frac{\partial U_m}{\partial x_l}-\sqrt{2D}\frac{\partial U_m}{\partial x_l}\times\int_0^{t_2-t_1}w_l ds+\sqrt{2D}w_m(t_2-t_1)+\cdots. \tag{A8}$$

Combination Eqs. (A8) and (A5) yields the field $n_p(t,\vec{\xi}_{\Delta t})$

$$n_p(t,\vec{\xi}_{\Delta t})=n_p(t,\mathbf{x})+\frac{\partial n_p}{\partial x_m}\left[-U_m\Delta t+\frac{1}{2}U_l\frac{\partial U_m}{\partial x_l}(\Delta t)^2+\sqrt{2D}w_m-\sqrt{2D}\frac{\partial U_m}{\partial x_l}\int_0^{\Delta t}w_l ds\right]+\frac{1}{2}\frac{\partial^2 n_p}{\partial x_m\partial x_s}[U_mU_s(\Delta t)^2+2Dw_mw_s-\sqrt{2D}\Delta t(U_mw_s+U_sw_m)], \tag{A9}$$

Here we keep the terms up to $\geqslant O[(\Delta t)^2]$.

(iii) Now we expand the function $b_*[\sigma,\vec{\xi}(t+\Delta t,\sigma)]$ in the Taylor series in the vicinity of point $\mathbf{x}$, and calculate the integral

$$\int_t^{t+\Delta t}b_*[\sigma,\vec{\xi}(t+\Delta t,\sigma)]d\sigma.$$

The result is given by

$$\int_t^{t+\Delta t}b_*(\sigma,\vec{\xi}_\sigma)d\sigma\simeq b_*(t,\mathbf{x})\Delta t-\tfrac{1}{2}U_q\frac{\partial b_*}{\partial x_q}(\Delta t)^2+\sqrt{2D}\frac{\partial b_*}{\partial x_q}\int_T^{T+\Delta t}w_q d\sigma+\cdots. \tag{A10}$$

Here we also keep terms $\geqslant O[(\Delta t)^2]$. Using Eq. (A10) we calculate the function $G(t+\Delta t,t)$ accurate up to $\sim O[(\Delta t)^2]$:

$$G(t+\Delta t,t)\simeq 1-b_*(t,\mathbf{x})\Delta t+\tfrac{1}{2}U_q\frac{\partial b_*}{\partial x_q}(\Delta t)^2+\tfrac{1}{2}b_*^2(\Delta t)^2-\sqrt{2D}\frac{\partial b_*}{\partial x_q}\int_t^{t+\Delta t}w_q d\sigma. \tag{A11}$$

(iv) Substituting Eq. (A11) and (A9) into Eq. (A4) allows us to determine the number density $n_p(t+\Delta t,\mathbf{x})$:

$$n_p(t+\Delta t,\mathbf{x})=M\left\{n_p(t,\mathbf{x})+n_1(\Delta t)+n_2(\Delta t)^2+D\frac{\partial^2 n_p}{\partial x_m\partial x_l}w_mw_s\right\}, \tag{A12}$$

where

$$n_1=U_m\frac{\partial n_p}{\partial x_m}-b_*n_p,$$

$$n_2=\frac{\partial n_p}{\partial x_m}\left(\frac{1}{2}U_l\frac{\partial U_m}{\partial x_l}+b_*U_m\right)+\frac{1}{2}n_p\left(U_l\frac{\partial b_*}{\partial x_l}+b_*^2\right)+\frac{1}{2}\frac{\partial^2 n_p}{\partial x_m\partial x_l}U_mU_l.$$

Note that the velocity $\mathbf{U}$ is determined by the turbulent velocity $\mathbf{v}$ of the surrounding fluid [see Eq. (30)]. In order to determine the mean field N we average Eq. (A12) for the number density $n_p(t+\Delta t,\mathbf{x})$ over the turbulent velocity $\mathbf{U}$, (i.e. $N=\langle n\rangle$). Note that $\mathbf{U}=\mathbf{V}_p+\mathbf{u}$, where $\mathbf{V}_p=\langle\mathbf{U}\rangle$ is the mean velocity and $\mathbf{u}$ is the random component of the velocity of particles. It is important to note that the Wiener random process $\mathbf{w}(t)$ and the turbulent velocity $\mathbf{u}(t,\mathbf{x})$ are independent random processes, and therefore we can change the order of averaging: $\langle M\{f\}\rangle\to M\{\langle f\rangle\}$ (see, e.g., [8,9]). On the other hand, the random processes $\mathbf{w}(t)$ and $\mathbf{u}(t,\vec{\xi}_{\Delta t})$ are correlated. We also assume that the velocities $\mathbf{u}$ in both intervals $(0,t)$ and $(t,t+\Delta t)$ are independent, because we consider the random flow with a short time of renewal. Note that averaging over the Wiener paths corresponds to the averaging over the molecular processes with very small characteristic scales. On the other hand, $\langle f\rangle$ determines the averaging over the turbulent velocity field with scales that are larger than the molecular ones.

(v) Now we calculate

$$\frac{N(t+\Delta t,\mathbf{x})-N(t,\mathbf{x})}{\Delta t},$$

and pass to the limit $\Delta t\to 0$. Here $N=\langle n_p\rangle$. The result is given by

$$\frac{\partial N}{\partial t}+[(\mathbf{V}-\langle\tau(\mathbf{u}\cdot\vec{\nabla})\mathbf{u}\rangle-2\langle\tau b\mathbf{u}\rangle)\cdot\vec{\nabla}]N=B_{\rm eff}N+D_{pm}\frac{\partial^2 N}{\partial x_p\partial x_m}, \tag{A13}$$

where $B_{\rm eff}=-(\vec{\boldsymbol{\nabla}}\cdot\mathbf{V})+\langle\tau(\mathbf{u}\cdot\vec{\boldsymbol{\nabla}})b\rangle+\langle\tau b^2\rangle$. Using the identity

$$\left\langle \tau u_p \frac{\partial}{\partial x_p} u_m \right\rangle = \frac{\partial}{\partial x_p}\langle \tau u_p u_m\rangle - \langle \tau \mathbf{u}(\vec{\boldsymbol{\nabla}}\cdot\mathbf{u})\rangle$$

we obtain Eq. (32) for the mean number density of particles.

(vi) Now we derive equation for the second-order correlation function for the number density of particles. Let us calculate the correlation function $\Phi(t+\Delta t,\mathbf{x},\mathbf{y})=\langle\Theta(t+\Delta t,\mathbf{x})\Theta(t+\Delta t,\mathbf{y})\rangle$ by means of Eq. (A12). The obtained equation allows us to find the function

$$\frac{\Phi(t+\Delta t)-\Phi(t)}{\Delta t}.$$

Passing to the limit $\Delta t \to 0$ yields, Eq. (2) for the correlation function Φ.

[1] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **76**, 224 (1996).
[2] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **55**, 2713 (1997).
[3] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **77**, 5373 (1996).
[4] J. R. Fessler, J. D. Kulick, and J. K. Eaton, Phys. Fluids **6**, 3742 (1994).
[5] P. V. Hobbs and A. L. Rangno, J. Atmos. Sci. **37**, 2486 (1985).
[6] A. V. Korolev and I. P. Mazin, J. Appl. Meteorol. **32**, 760 (1993).
[7] Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokoloff, Zh. Éksp. Teor. Fiz. **89**, 2061 (1985) [Sov. Phys. JETP **62**, 1188 (1985)]
[8] Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokoloff, in *Mathematical Physics Reviews*, edited by S. P. Novikov, Soviet Scientific Reviews, Section C (Harwood Academic, Chur, Switzerland, 1988), Vol. 7, p. 1, and references therein.
[9] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *The Almighty Chance* (Word Scientific, London, 1990), and references therein.
[10] I. Rogachevskii and N. Kleeorin, Phys. Rev. E **56**, 417 (1997).
[11] R. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[12] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995); M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1996); M. Chertkov, G. Falkovich, and V. Lebedev, *ibid.* **76**, 3707 (1996).
[13] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).
[14] B. I. Shraiman and E. D. Siggia, C. R. Acad. Sci., Ser. I: Math. **321**, 279 (1995).
[15] M. Vergassola, Phys. Rev. E **53**, 3021 (1996).
[16] P. Dittrich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokoloff, Astron. Nachr. **305**, 119 (1984).
[17] N. Kleeorin and I. Rogachevskii, Phys. Rev. E **50**, 493 (1994).
[18] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **55**, 7043 (1997).
[19] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[20] M. R. Maxey, J. Fluid Mech. **174**, 441 (1987).
[21] M. R. Maxey and J. J. Riley, Phys. Fluids **26**, 883 (1983).
[22] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).
[23] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **52**, 2617 (1995).
[24] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **80**, 69 (1998).
[25] P. C. Reist, *Aerosol Science and Technology* (McGraw-Hill, New York, 1993).
[26] M. Caporaloni, F. Tampieri, F. Trombetti, and O. Vittori, J. Atmos. Sci. **32**, 565 (1975).
[27] M. W. Reeks, J. Aerosol Sci. **14**, 729 (1983).
[28] Z. Schuss, *Theory and Applications of Stochastic Differential Equations* (Wiley, New York, 1980).